Similarly, $JI'' \perp CA$ and $JI''' \perp AB$.

(4) Arc $P''B =$ arc $P'C$. Hence, $\angle P''PB = \angle P'PC$; hence $\angle BPR = \angle RPC$. Hence, $BR : RC = BP : PC$. Similarly, $CR' : R'A = CP' : P'A$, and $AR'' : R''B = AP'' : P''B$. Hence,

$$\frac{BR \cdot CR' \cdot AR''}{RC \cdot R'A \cdot R''B} = \frac{BP \cdot CP' \cdot AP''}{PC \cdot P'A \cdot P''B}.$$

But $BP = AP'$, being opposite sides of a rectangle. Similarly, $CP' = BP''$ and $AP'' = CP$. Hence,

$$\frac{BR \cdot CR' \cdot AR''}{RC \cdot R'A \cdot R''B} = 1,$$

numerically. Hence, AR, BR', CR'' are concurrent.

Also solved by A. M. Harding.

CALCULUS.

373. Proposed by C. N. SCHMALL, New York City.

In the *Encyclopaedia Britannica* article on "Capillary Action" (Vol. 5, p. 268, 11th ed.) it is shown that $1/R_1 + 1/R_2 = p/T$, in the case of a soap bubble, where R_1, R_2 are the *principal radii of curvature* at any point of the bubble; p, the difference of air-pressure; T, the energy per unit area of the film. Employing the principle that the soap bubble tends to assume a form such that the area of its surface is a *minimum* for a *given volume* of air, show by the calculus of variations that $1/R_1 + 1/R_2 = k$, a constant.

Solution by the Proposer.

We have here to determine the solid which, with a given volume (capacity), contains the least surface. Hence, we have to make the surface integral

$$U = \int\int \sqrt{1 + p^2 + q^2}\, dxdy \tag{1}$$

a minimum, subject to the condition that the volume integral

$$I = \int\int z\, dxdy \tag{2}$$

is constant.

It should be remembered that in this discussion $p = \partial z/\partial x$, $q = \partial z/\partial y$, $r = \partial^2 z/\partial x^2$, $s = \partial^2 z/\partial x \partial y$, and $t = \partial^2 z/\partial y^2$.

Let k be a constant. Then it is evident that, when the surface (1) is a minimum, the binomial

$$\begin{aligned}\int\int \sqrt{1+p^2+q^2}\, dxdy + \int\int kz\, dxdy &\equiv \int\int (\sqrt{1+p^2+q^2} + kz)dxdy \\ &\equiv \int\int V\, dxdy\end{aligned} \tag{3}$$

will also be a minimum.

Now, taking x and y as the independent variables, the condition for a mini-

mum is, by the calculus of variations,

$$\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = L, \tag{4}$$

(TODHUNTER'S *Int. Cal.*, p. 374, 5th ed.)

where

$$M = \frac{\partial V}{\partial p} = \frac{p}{\sqrt{1 + p^2 + q^2}},$$

$$N = \frac{\partial V}{\partial q} = \frac{q}{\sqrt{1 + p^2 + q^2}},$$

$$L = \frac{\partial V}{\partial z} = k.$$

(TODHUNTER'S, *Int. Cal.*, p. 372.)

Hence, equation (4) becomes

$$\frac{\partial}{\partial x}\left(\frac{p}{\sqrt{1 + p^2 + q^2}}\right) + \frac{\partial}{\partial y}\left(\frac{q}{\sqrt{1 + p^2 + q^2}}\right) = k \tag{5}$$

or,

$$\frac{r(1 + p^2 + q^2) - (pr + qs)p + t(1 + p^2 + q^2) - (ps + qt)q}{(1 + p^2 + q^2)^{\frac{3}{2}}} = k,$$

or,

$$\frac{(1 + q^2)r - 2pqs + (1 + p^2)t}{(1 + p^2 + q^2)^{\frac{3}{2}}} = k, \tag{6}$$

which is the partial differential equation of the required minimal surface, the integral of which will represent the surface itself.

Again, R_1 and R_2 are known as the *principal radii of curvature* at any point of the surface. The equation giving these is

$$(rt - s^2)\,R^2 - \sqrt{1 + p^2 + q^2}\,[(1 + p^2)t - 2pqs + (1 + q^2)r]R + (1 + p^2 + q^2)^2 = 0. \tag{7}$$

(GOURSAT-HEDRICK'S *Math. Anal.*, Vol. 1, p. 504, Eq. 13.)

If R_1, R_2, be the roots of this quadratic in R, we have

$$R_1 + R_2 = \frac{\sqrt{1 + p^2 + q^2}[(1 + p^2)t - 2pqs + (1 + q^2)r]}{rt - s^2}, \tag{8}$$

$$R_1R_2 = \frac{(1 + p^2 + q^2)^2}{rt - s^2}. \tag{9}$$

Dividing (8) by (9), we obtain

$$\frac{1}{R_1} + \frac{1}{R_2} = \frac{(1 + p^2)t - 2pqs + (1 + q^2)r}{(1 + p^2 + q^2)^{\frac{3}{2}}}. \tag{10}$$

(See EISENHART'S *Diff. Geom.*, p. 126, ex. 3.)

Comparing equations (6) and (10) we have the required result,

$$\frac{1}{R_1} + \frac{1}{R_2} = k.$$

Note 1.—The sphere,

$$x^2 + y^2 + z^2 = a^2 \tag{11}$$

and the cylinder,

$$y^2 + z^2 = b^2, \tag{12}$$

are examples of minimal surfaces satisfying Eq. (6).

Thus, in (11),

$$p = -x/z, \qquad q = -y/z, \qquad \sqrt{1+p^2+q^2} = a/z,$$

$$\therefore \quad M = -x/a, \qquad N = -y/a,$$

and (11) becomes $k + 2/a = 0$; and (12), $k + 1/b = 0$.

Note 2.—In the foregoing solution I have utilized the notation employed in the chapter on the Calculus of Variations in TODHUNTER's *Integral Calculus*, fifth edition.

374. Proposed by C. N. SCHMALL, New York City.

Show that, on a *Mercator's Chart*, a great circle of a sphere of radius r_1 will be represented by a curve whose equation is of the form

$$c(e^{y/r} - e^{-(y/r)}) = 2 \sin\left(\frac{x}{r} + \theta\right).$$

I. SOLUTION BY ELIJAH SWIFT, University of Vermont.

If the latitude and longitude on the above sphere be denoted by the letters φ and θ respectively, θ varying from 0° to 360°, and φ from $-90°$ to $+90°$; then if axes be taken with origin at the center of the sphere with the xy-plane as the plane of the equator, and if longitude be measured from the x-axis, we have for any point on the sphere

$$x = r \cos\varphi \cos\theta, \; y = r \cos\varphi \sin\theta, \; z = r \sin\varphi.$$

The equation of a great circle is obtained by substituting these values in the equation of any diametral plane, $Ax + By + Cz = 0$, and is

$$A \cos\varphi \cos\theta + B \cos\varphi \sin\theta + C \sin\varphi = 0. \tag{1}$$

The sphere is mapped on a *Mercator's Chart* by taking a cylinder tangent to the sphere along the equator and projecting a meridian ($\theta =$ const.) on a generating line of the cylinder.

Any point on the sphere on this meridian has for its image on the chart a point on the corresponding generating line at a distance $r \log \tan (\pi/4 + \varphi/2)$ from the equator.

When we develop the cylinder on a plane, we can choose axes in that plane so that the coördinates of this point are

$$x = r\theta, \qquad y = r \log \tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right).$$

Solving these equations for θ and φ, $\theta = x/r$, $\varphi = 2 \arctan (e^{y/r}) - \pi/2$. Substituting these values in (1), we obtain

$$A \cos\frac{x}{r} + B \sin\frac{x}{r} + C \tan\left\{2 \arctan e^{y/r} - \frac{\pi}{2}\right\} = 0,$$

which reduces at once to the form given, if we let $c = -C/\sqrt{A^2+B^2}$, and $\sin\theta = A/\sqrt{A^2+B^2}$.